➤ Problems similar to Example 2 were solved in Section 12.8 using ordinary optimization techniques. These methods may or may not be easier to apply than Lagrange multipliers.

EXAMPLE 2 A geometry problem Find the least distance between the point $P(3, 4, 0)$ and the surface of the cone $z^2 = x^2 + y^2$.

SOLUTION Figure 12.100 shows both sheets of the cone and the point $P(3, 4, 0)$. Because P is in the xy-plane, we anticipate two solutions, one for each sheet of the cone. The distance between P and any point $Q(x, y, z)$ on the cone is

$$d(x, y, z) = \sqrt{(x-3)^2 + (y-4)^2 + z^2}.$$

In many distance problems it is easier to work with the *square* of the distance to avoid dealing with square roots. This maneuver is allowable because if a point minimizes $(d(x, y, z))^2$, it also minimizes $d(x, y, z)$. Therefore, we define

$$f(x, y, z) = (d(x, y, z))^2 = (x-3)^2 + (y-4)^2 + z^2.$$

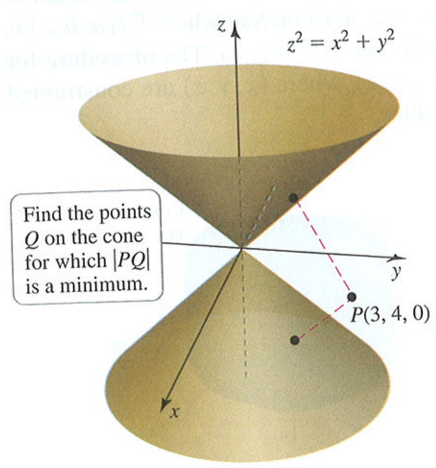


FIGURE 12.100

The constraint is the condition that the point (x, y, z) must lie on the cone, which implies $z^2 = x^2 + y^2$, or $g(x, y, z) = z^2 - x^2 - y^2 = 0$.

Now we proceed with Lagrange multipliers; the conditions are

$$f_x(x, y, z) = \lambda g_x(x, y, z), \text{ or } 2(x-3) = \lambda(-2x), \text{ or } x(1+\lambda) = 3 \quad (4)$$
$$f_y(x, y, z) = \lambda g_y(x, y, z), \text{ or } 2(y-4) = \lambda(-2y), \text{ or } y(1+\lambda) = 4 \quad (5)$$
$$f_z(x, y, z) = \lambda g_z(x, y, z), \text{ or } 2z = \lambda(2z), \text{ or } z = \lambda z \quad (6)$$
$$g(x, y, z) = z^2 - x^2 - y^2 = 0. \quad (7)$$

The solutions of equation (6) (the simplest of the four equations) are either $z = 0$, or $\lambda = 1$ and $z \neq 0$. In the first case, if $z = 0$, then by equation (7), $x = y = 0$; however, $x = 0$ and $y = 0$ do not satisfy (4) and (5). So no solution results from this case.

On the other hand if $\lambda = 1$, then by (4) and (5), we find that $x = \frac{3}{2}$ and $y = 2$. Using (7), the corresponding values of z are $\pm\frac{5}{2}$. Therefore, the two solutions and the values of f are

➤ With three independent variables, it is possible to impose two constraints. These problems are explored in Exercises 53–57.

$$x = \tfrac{3}{2}, \quad y = 2, \quad z = \tfrac{5}{2} \quad \text{with } f\left(\tfrac{3}{2}, 2, \tfrac{5}{2}\right) = \tfrac{25}{2}$$
$$x = \tfrac{3}{2}, \quad y = 2, \quad z = -\tfrac{5}{2} \quad \text{with } f\left(\tfrac{3}{2}, 2, -\tfrac{5}{2}\right) = \tfrac{25}{2}$$

You can check that moving away from $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$ in any direction on the cone has the effect of increasing the values of f. Therefore, the points correspond to *local* minima of the distance function. Do these points also correspond to *absolute* minima? The domain of this problem is unbounded; however, one can argue geometrically that the distance function increases without bound moving away from $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$. Therefore, these points correspond to absolute minimum values and the points on the cone nearest to $(3, 4, 0)$ are $\left(\frac{3}{2}, 2, \pm\frac{5}{2}\right)$, at a distance of $\sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$.

Related Exercises 11–26 ◄

QUICK CHECK 3 In Example 2, is there a point that *maximizes* the distance between $(3, 4, 0)$ and the cone? If the point $(3, 4, 0)$ were replaced by $(3, 4, 1)$, how many minimizing solutions would there be? ◄

Economic Models In the opening of this section, we briefly described how utility functions are used to model consumer behavior. We now look in more detail at some specific—admittedly simple—utility functions and the constraints that are imposed upon them.

As described earlier, a prototype model for consumer behavior uses two independent variables: leisure time ℓ and consumable goods g. A utility function $U = f(\ell, g)$ measures consumer preferences for various combinations of leisure time and consumable goods. The following assumptions about utility functions are commonly made:

1. Utility increases if any variable increases (essentially, *more is better*).
2. Various combinations of leisure time and consumable goods have the same utility; that is, giving up some leisure time for additional consumable goods results in the same utility.

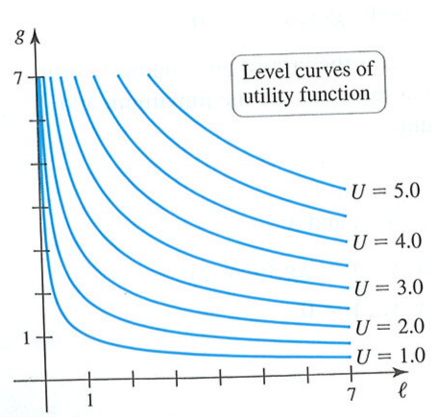


FIGURE 12.101

The level curves of a typical utility function are shown in Figure 12.101. Assumption 1 is reflected by the fact that the utility values on the level curves increase as either ℓ or g increases. Consistent with Assumption 2, a single level curve shows the combinatio[ns] of ℓ and g that have the same utility; for this reason, economists call the level curv[es] *indifference curves*. Notice that if ℓ increases, then g must decrease on a level curve [to] maintain the same utility, and vice versa.

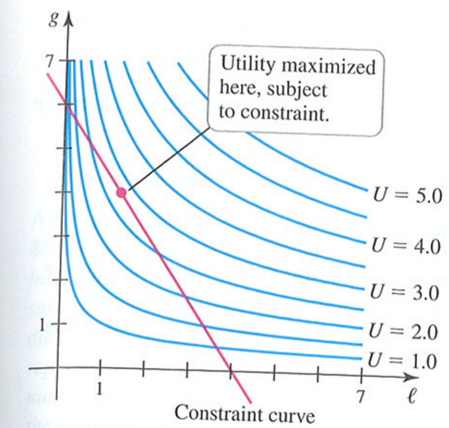


FIGURE 12.102

Economic models assert that consumers maximize utility subject to constraints [on] leisure time and consumable goods. One assumption that leads to a reasonable constrai[nt] is that an increase in leisure time implies a linear decrease in consumable goods. Ther[e]fore, the constraint curve is a line with negative slope (Figure 12.102). When such a co[n]straint is superimposed on the level curves of the utility function, the optimizati[on] problem becomes evident. Among all points on the constraint line, which one maximiz[es] utility? A solution is marked in the figure; at this point the utility has a maximum valu[e] (between 2.5 and 3.0).

EXAMPLE 3 Constrained optimization of utility Find the maximum value of the utility function $U = f(\ell, g) = \ell^{1/3}g^{2/3}$, subject to the constraint $G(\ell, g) = 3\ell + 2g - 12 = 0$, where $\ell \geq 0$ and $g \geq 0$.

SOLUTION The level curves of the utility function and the linear constraint are shown in Figure 12.102. The solution follows the Lagrange multiplier method with two variables. The gradient of the utility function is

$$\nabla f(\ell, g) = \left\langle \frac{\ell^{-2/3}g^{2/3}}{3}, \frac{2\ell^{1/3}g^{-1/3}}{3} \right\rangle = \frac{1}{3}\left\langle \left(\frac{g}{\ell}\right)^{2/3}, 2\left(\frac{\ell}{g}\right)^{1/3} \right\rangle.$$

The gradient of the constraint function is $\nabla G(\ell, g) = \langle 3, 2 \rangle$. Therefore, the equations that must be solved are

$$\frac{1}{3}\left(\frac{g}{\ell}\right)^{2/3} = 3\lambda, \quad \frac{2}{3}\left(\frac{\ell}{g}\right)^{1/3} = 2\lambda, \quad G(\ell, g) = 3\ell + 2g - 12 = 0.$$

QUICK CHECK 4 In Figure 12.102, explain why, if you move away from the optimal point along the constraint line, the utility decreases. ◄

Eliminating λ from the first two equations leads to the condition $g = 3\ell$, which, when substituted into the constraint equation, gives the solution $\ell = \frac{4}{3}$ and $g = 4$. The actual value of the utility function at this point is $U = f\left(\frac{4}{3}, 4\right) = 4/\sqrt[3]{3} \approx 2.8$. This solution is consistent with Figure 12.102.

Related Exercises 27–30 ◄

SECTION 12.9 EXERCISES

Review Questions

1. Explain why, at a point that maximizes or minimizes f subject to a constraint $g(x, y) = 0$, the gradient of f is parallel to the gradient of g. Use a diagram.
2. If $f(x, y) = x^2 + y^2$ and $g(x, y) = 2x + 3y - 4 = 0$, write the Lagrange multiplier conditions that must be satisfied by a point that maximizes or minimizes f subject to $g(x, y) = 0$.
3. If $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = 2x + 3y - 5z + 4 = 0$, write the Lagrange multiplier conditions that must be satisfied by a point that maximizes or minimizes f subject to $g(x, y, z) = 0$.
4. Sketch several level curves of $f(x, y) = x^2 + y^2$ and sketch the constraint line $g(x, y) = 2x + 3y - 4 = 0$. Describe the extrema (if any) that f attains on the constraint line.

Basic Skills

5–10. Lagrange multipliers in two variables *Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint.*

5. $f(x, y) = x + 2y$ subject to $x^2 + y^2 = 4$
6. $f(x, y) = xy^2$ subject to $x^2 + y^2 = 1$
7. $f(x, y) = e^{2xy}$ subject to $x^3 + y^3 = 16$
8. $f(x, y) = x^2 + y^2$ subject to $x^6 + y^6 = 1$
9. $f(x, y) = y^2 - 4x^2$ subject to $x^2 + 2y^2 = 4$
10. $f(x, y) = xy + x + y$ subject to $xy = 4$

11–16. Lagrange multipliers in three or more variables *Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint.*

11. $f(x, y, z) = x + 3y - z$ subject to $x^2 + y^2 + z^2 = 4$
12. $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$
13. $f(x, y, z) = xy^2z^3$ subject to $x^2 + y^2 + 2z^2 = 25$
14. $f(x, y, z) = x^2 + y^2 + z^2$ subject to $z = 1 + 2xy$
15. $f(x, y, z) = x^2 + y^2 + z^2$ subject to $xyz = 4$

16. $f(x, y, z) = (xyz)^{1/2}$ subject to $x + y + z = 1$ with $x \ge 0$, $y \ge 0, z \ge 0$

17–26. Applications of Lagrange multipliers *Use Lagrange multipliers in the following problems. When the domain of the objective function is unbounded or open, explain why you have found an absolute maximum or minimum value.*

17. **Shipping regulations** A shipping company requires that the sum of length plus girth of rectangular boxes must not exceed 108 in. Find the dimensions of the box with maximum volume that meets this condition. (The girth is the perimeter of the smallest base of the box.)

18. **Box with minimum surface area** Find the rectangular box with a volume of 16 ft^3 that has minimum surface area.

T 19. **Extreme distances to an ellipse** Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

20. **Maximum area rectangle in an ellipse** Find the dimensions of the rectangle of maximum area with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 16y^2 = 16$.

21. **Maximum perimeter rectangle in an ellipse** Find the dimensions of the rectangle of maximum perimeter with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$.

22. **Minimum distance to a plane** Find the point on the plane $2x + 3y + 6z - 10 = 0$ closest to the point $(-2, 5, 1)$.

T 23. **Minimum distance to a surface** Find the point on the surface $x^2 - 2xy + 2y^2 - x + y = 0$ closest to the point $(1, 2, -3)$.

24. **Minimum distance to a cone** Find the points on the cone $z^2 = x^2 + y^2$ closest to the point $(1, 2, 0)$.

25. **Extreme distances to a sphere** Find the minimum and maximum distances between the sphere $x^2 + y^2 + z^2 = 9$ and the point $(2, 3, 4)$.

26. **Maximum volume cylinder in a sphere** Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 16.

27–30. Maximizing utility functions *Find the values of ℓ and g with $\ell \ge 0$ and $g \ge 0$ that maximize the following utility functions subject to the given constraints. Give the value of the utility function at the optimal point.*

27. $U = f(\ell, g) = 10\ell^{1/2}g^{1/2}$ subject to $3\ell + 6g = 18$

28. $U = f(\ell, g) = 32\ell^{2/3}g^{1/3}$ subject to $4\ell + 2g = 12$

29. $U = f(\ell, g) = 8\ell^{4/5}g^{1/5}$ subject to $10\ell + 8g = 40$

30. $U = f(\ell, g) = \ell^{1/6}g^{5/6}$ subject to $4\ell + 5g = 20$

Further Explorations

31. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

   a. Suppose you are standing at the center of a sphere looking at a point P on the surface of the sphere. Your line of sight to P is orthogonal to the plane tangent to the sphere at P.

   b. At a point that maximizes f on the curve $g(x, y) = 0$, the dot product $\nabla f \cdot \nabla g$ is zero.

32–37. *Solve the following problems from Section 12.8 using Lagrange multipliers.*

32. Exercise 29	33. Exercise 30	34. Exercise 31
35. Exercise 32	36. Exercise 56	37. Exercise 57

T **38–41. Absolute maximum and minimum values** *Find the absolute maximum and minimum values of the following functions over the given regions R. Use Lagrange multipliers to check for extreme points on the boundary.*

38. $f(x, y) = x^2 + 4y^2 + 1$; $R = \{(x, y): x^2 + 4y^2 \le 1\}$

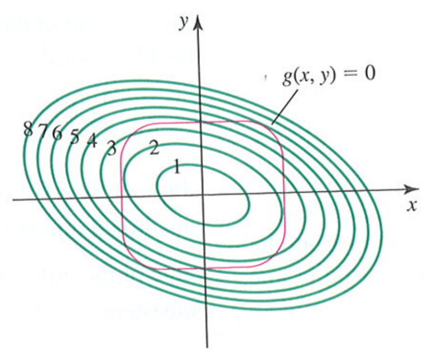
39. $f(x, y) = x^2 - 4y^2 + xy$; $R = \{(x, y): 4x^2 + 9y^2 \le 36\}$

40. $f(x, y) = 2x^2 + y^2 + 2x - 3y$; $R = \{(x, y): x^2 + y^2 \le 1\}$

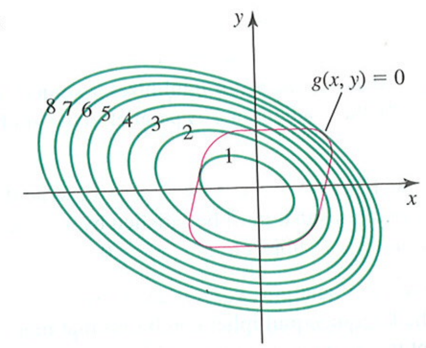
41. $f(x, y) = (x - 1)^2 + (y + 1)^2$; $R = \{(x, y): x^2 + y^2 \le 4\}$

42–43. Graphical Lagrange multipliers *The following figures show the level curves of f and the constraint curve $g(x, y) = 0$. Estimate the maximum and minimum values of f subject to the constraint. At each point where an extreme value occurs, indicate the direction of ∇f and a possible direction of ∇g.*

42.



43.



44. **Extreme points on flattened spheres** The equation $x^{2n} + y^{2n} + z^{2n} = 1$, where n is a positive integer, describes a flattened sphere. Define the extreme points to be the points on the flattened sphere with a maximum distance from the origin.

   a. Find all the extreme points on the flattened sphere with $n = 2$. What is the distance between the extreme points and the origin?

   b. Find all the extreme points on the flattened sphere for integers $n > 2$. What is the distance between the extreme points and the origin?

   c. Give the location of the extreme points in the limit as $n \to \infty$. What is the limiting distance between the extreme points and the origin as $n \to \infty$?

Applications

45–47. Production functions *Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form $P = f(K, L) = CK^aL^{1-a}$, where K represents capital, L represents labor, and C and a are positive real numbers with $0 < a < 1$. If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form $pK + qL = B$. Find the values of K and L that maximize the following production functions subject to the given constraint, assuming $K \ge 0$ and $L \ge 0$.*

45. $P = f(K, L) = K^{1/2}L^{1/2}$ for $20K + 30L = 300$

46. $P = f(K, L) = 10K^{1/3}L^{2/3}$ for $30K + 60L = 360$

47. Given the production function $P = f(K, L) = K^aL^{1-a}$ and the budget constraint $pK + qL = B$, where a, p, q, and B are given, show that P is maximized when $K = aB/p$ and $L = (1 - a)B/q$.

48. **Least squares approximation** Find the coefficients in the equation of the plane $z = ax + by + c$ that minimize the sum of the squares of the vertical distances between the plane and the points $(1, 2, 3)$, $(-2, 3, 1)$, $(3, 0, -4)$, and $(0, -2, 6)$.

Additional Exercises

49–51. Maximizing a sum

49. Find the maximum value of $x_1 + x_2 + x_3 + x_4$ subject to the condition that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$.

50. Generalize Exercise 49 and find the maximum value of $x_1 + x_2 + \cdots + x_n$ subject to the condition that $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$ for a real number c and a positive integer n.

51. Generalize Exercise 49 and find the maximum value of $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ subject to the condition that $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ for given positive real numbers $a_1, \ldots, a_n$ and a positive integer n.

52. **Geometric and arithmetic means** Prove that the geometric mean of a set of positive numbers $(x_1x_2\cdots x_n)^{1/n}$ is no greater than the arithmetic mean $(x_1 + \cdots + x_n)/n$ in the following cases.

   a. Find the maximum value of xyz, subject to $x + y + z = k$, where k is a real number and $x > 0$, $y > 0$, and $z > 0$. Use the result to prove that

$$(xyz)^{1/3} \le \frac{x + y + z}{3}$$

   b. Generalize part (a) and show that

$$(x_1x_2\cdots x_n)^{1/n} \le \frac{x_1 + \cdots + x_n}{n}.$$

53. **Problems with two constraints** Given a differentiable function $w = f(x, y, z)$, the goal is to find its maximum and minimum values subject to the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, where g and h are also differentiable.

   a. Imagine a level surface of the function f and the constraint surfaces $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Note that g and h intersect (in general) in a curve C on which maximum and minimum values of f must be found. Explain why ∇g and ∇h are orthogonal to their respective surfaces.

   b. Explain why ∇f lies in the plane formed by ∇g and ∇h at a point of C where f has a maximum or minimum value.

   c. Explain why part (b) implies that $\nabla f = \lambda \nabla g + \mu \nabla h$ at a point of C where f has a maximum or minimum value, where λ and μ (the Lagrange multipliers) are real numbers.

   d. Conclude from part (c) that the equations that must be solved for maximum or minimum values of f subject to two constraints are $\nabla f = \lambda \nabla g + \mu \nabla h$, $g(x, y, z) = 0$, and $h(x, y, z) = 0$.

54–57. Two-constraint problems *Use the result of Exercise 53 to solve the following problems.*

54. The planes $x + 2z = 12$ and $x + y = 6$ intersect in a line L. Find the point on L nearest the origin.

55. Find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the conditions that $x^2 + y^2 = 4$ and $x + y + z = 1$.

56. The paraboloid $z = x^2 + 2y^2 + 1$ and the plane $x - y + 2z = 4$ intersect in a curve C. Find the points on C that have minimum and maximum distance from the origin.

57. Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ on the intersection between the cone $z^2 = 4x^2 + 4y^2$ and the plane $2x + 4z = 5$.

QUICK CHECK ANSWERS

1. Let $d(x, y)$ be the distance between any point $P(x, y)$ on the fence and home plate O. The key fact is that ∇d always points along the line OP. As P moves along the fence (the constraint curve), $d(x, y)$ increases until a point is reached at which ∇d is orthogonal to the fence. At this point, d has a maximum value. 3. The distance between $(3, 4, 0)$ and the cone can be arbitrarily large, so there is no maximizing solution. If the point of interest is not in the xy-plane, there is one minimizing solution. 4. If you move along the constraint line away from the optimal solution in either direction, you cross level curves of the utility function with decreasing values. ◄